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the deviation in one cycle and multiplying by 8^{y}, we find for the period of change through one interval of the cycle, $235^{\text{y}}\cdot 182$ (sidereal years).

It would be quite possible in this way to calculate the dates at which the observations of this tablet could have been made; but a conjectural element enters into the reconstruction of the calendar of the observations. And as there is nothing to associate these observations with historical dates, there is no possibility of a real contribution to ancient history in this case.

It is, however, amply proved that these observations do actually refer to *Venus*, except the portion which we have excepted as spurious. It is further shown that the writer of the tablet, which has come down to us from remote antiquity, was himself an ignorant copyist of an earlier tablet or tablets; but of what date are the observations of this earlier tablet or tablets there is no evidence to show, except the antique style, and the fact of their belonging to the collection supposed to have been made by Sargon of Aganè; which tends to refer them to a period older than 1700 years B.C.

The length to which this comment has extended, and other circumstances, compel us to postpone the discussion of the identification of stars, &c.

Addition to a Paper entitled "On the Theoretical Value of the Acceleration of the Moon's Mean Motion in Longitude produced by the Change of Excentricity of the Earth's Orbit." By Sir G. B. Airy, K.C.B., Astronomer Royal.

In a paper lately communicated to the Society, my principal object was to exhibit the power of a new or factorial method of obtaining the disturbance of the Moon's movements depending on the introduction or change of an external force; and I applied it to the disturbance called the "Acceleration of the Moon's Mean Motion," produced by a gradual change in the excentricity of the earth's orbit. In completing the calculation, and estimating for that purpose the external factor of the formula, I limited myself to the imperfect expressions for the change of magnitude of the force, which (historically) had been adopted in the earliest investigations. I now propose to employ the more complete formula; referring, when advantageous, to the preceding paper, and using it to abbreviate this communication where it appears possible to do so.

(19.) The Sun's disturbing action is to be treated here in the same form as in the preceding articles; employing the symbol T for the ecliptic tangential accelerating force, and P for the ecliptic radial force measured from the Earth, and using V and v for the true longitudes of the Sun and Moon, R for the Sun's radius vector, and r for the Moon's radius vector, A for

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the semi-axis major of the Sun's apparent orbit, and a for the reciprocal of the non-periodic term in the reciprocal of the Moon's radius vector. The units of measure have been explained in the preceding part of the paper. Then—

$$\mathrm{T}\,.\,\frac{1}{a}\,.\,\frac{r}{a} = -{\cdot}0083928^{*}.\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^{3}.\left(\frac{r}{a}\right)^{2}.\sin\overline{|2(v-\mathrm{V})|}$$
$$-{\cdot}0000053\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^{4}.\left(\frac{r}{a}\right)^{3}.\sin\overline{|v-\mathrm{V}|}\,;$$

$$\mathrm{P}\,.\,\frac{1}{a}\,.\,\frac{r}{a} = +{\cdot}0083928^{*}.\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^{3}.\left(\frac{r}{a}\right)^{2}.\left\{+\frac{1}{3}+\cos\overline{|2(v-\mathrm{V})|}\right\}$$
$$-{\cdot}0000160\,.\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^{4}.\left(\frac{r}{a}\right)^{3}.\cos\overline{|v-\mathrm{V}|}\,;$$

with some very small terms depending on the argument $\overline{|3(v-\mathrm{V})|}$. These latter terms, as will be seen below, can only produce available results by being multiplied by other very small terms of nearly the same form, and the results will be far below our limits of accuracy for other terms. As regards cos $\overline{|v-\mathrm{V}|}$ in P (above), its factor is less than $\frac{1}{500}$ of the factor of cos $\overline{|2(v-\mathrm{V})|}$, and it will be multiplied by terms whose coefficients are less than $\frac{1}{10}$ of those of cos $\overline{|2(v-\mathrm{V})|}$, so that its efficient result will be less than $\frac{1}{5000}$ part. And that for T will be less than $\frac{1}{15000}$ part. I therefore reject these terms, and use only the following:

$$\mathrm{T}\,.\,\frac{1}{a}\,.\,\frac{r}{a} = -{\cdot}0083928\times\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^{3}.\left(\frac{r}{a}\right)^{2}\times\sin\overline{|2(v-\mathrm{V})|},$$

$$\mathrm{P}\,.\,\frac{1}{a}\,.\,\frac{r}{a} = +{\cdot}0083928\times\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^{3}.\left(\frac{r}{a}\right)^{2}\times\left\{+\frac{1}{3}+\cos\overline{|2(v-\mathrm{V})|}\right\}.$$

(20.) Before entering on the details of treatment of these formulæ, it will be convenient to premise the following remarks.

Our object is, to ascertain the effect produced in our numerical expressions, by a numerical change $\delta\mathrm{E}$ of the numerical quantity E, the excentricity of the solar orbit. It is necessary therefore to know what power of E has been employed in forming our numerical expressions. Now, E is always accompanied, in the first steps of expansion, by an argument containing S. It is true that E^3 or E^5 may have contributed in some instances (not in general) to form a portion of the coefficient of that argument containing S; but, if so, that portion of the coefficient must have received the numerical multiplier E^2 or E^4, either of which is extremely small, and which will have affected

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[*] By a fault of transcription, erroneous numbers are printed in the first investigation, p. 371. The term $-{\cdot}0077$ is not wanted here, being included in the further investigations.

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a very small part of the coefficient. Thus we may assume, without perceptible error, that the numerical coefficient of

$$\cos \lceil S \rceil, \quad \text{or} \quad \cos \lceil D-S \rceil, \ \&c.,$$

contains in itself the factor E; that the coefficient of

$$\cos \lceil 2S \rceil, \quad \text{or} \quad \cos \lceil 2D-2S \rceil, \quad \text{or} \quad \sin \lceil 2D-2S \rceil, \ \&c.,$$

contains in itself the factor E^2, and so on. (This, it will be observed, applies to the developments of R and V, but not always to more complicated functions, in which the trace of S may be destroyed by the union of two arguments; in these, the terms must be multiplied with due attention to their formation.) And, the power of E being thus determined, the variation of the coefficient, depending on δE, will be conveniently expressed thus. Let Q be a numerical coefficient containing the n^{th} power of E, so that $Q = W.E^n$ (W being a numeral, or a trigonometrical function, &c.). Then

$$\delta Q = n.W.E^{n-1}.\delta E = n.W.E^n.\frac{\delta E}{E} = n.Q.\frac{\delta E}{E}.$$

(21.) We shall now proceed to form the numerical value of $\left(\frac{A}{R}\right)^3.\left(\frac{r}{a}\right)^2$, with its variation depending on δE.

First, to form $\left(\frac{A}{R}\right)^3$,

Le Verrier gives for $\frac{R}{A}$,

$$+1{\cdot}0000000 \quad \overset{E^2}{+{\cdot}0001406} \quad \overset{E}{-{\cdot}0167687} . \cos \lceil S \rceil \quad \overset{E^2}{-{\cdot}0001406} . \cos \lceil 2S \rceil$$
$$\overset{E^3}{-{\cdot}0000018} . \cos \lceil 3S \rceil;$$

(in which the symbols E, E^2, E^3, placed above the figures, indicate the power of E from which those numbers have been formed): and from this, by ordinary expansion, we find for $\left(\frac{A}{R}\right)^3$,

$$+\left\{+1.0000000 \quad \overset{E^2}{+{\cdot}0004215}\right\} + \left\{\overset{E}{+{\cdot}0503061} \quad \overset{E^3}{+{\cdot}0000114}\right\} \cos \lceil S \rceil$$
$$\overset{E^2}{+{\cdot}0012651} . \cos \lceil 2S \rceil \quad \overset{E^3}{+0000309} . \cos \lceil 3S \rceil;$$

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and, forming the variation on the principles just described, and attaching it to the principal terms, we obtain for $\left(\frac{A}{R}\right)^3 + \delta \,.\, \left(\frac{A}{R}\right)^3$,

$$+1{\cdot}0004215 + {\cdot}0503275 \,.\, \cos \overline{|S|} + {\cdot}0012651 \,.\, \cos \overline{|2S|} + {\cdot}0000309 \,.\, \cos \overline{|3S|}$$

$$+\frac{\delta E}{E} \times \left\{ + {\cdot}0008430 + {\cdot}0503703 \,.\, \cos \overline{|S|} + {\cdot}0025302 \,.\, \cos \overline{|2S|} + {\cdot}0000927 \,.\, \cos \overline{|3S|} \right\}.$$

Secondly, to form $\left(\frac{r}{a}\right)^2$.

The expression which I have found by expansion from Delaunay's coefficients, including no arguments excepting those which contain D, S, and their combinations, is

$$+1{\cdot}0046872 + {\cdot}0005426 \,.\, \cos \overline{|D|} - {\cdot}0148320 \,.\, \cos \overline{|2D|} + \overset{E}{{\cdot}0002763 \,.\, \cos \overline{|S|}}$$

$$+ \overset{E^2}{{\cdot}0000075} \,.\, \cos \overline{|2S|} - \overset{E}{{\cdot}0000880} \,.\, \cos \overline{|D+S|} + \overset{E}{{\cdot}0000021} \,.\, \cos \overline{|D-S|}$$

$$+ \overset{E}{{\cdot}0001624} \,.\, \cos \overline{|2D+S|} - \overset{E}{{\cdot}0010217} \,.\, \cos \overline{|2D-S|} - \overset{E^2}{{\cdot}0000421} \,.\, \cos \overline{|2D-2S|}$$

$$- \overset{E}{{\cdot}0000021} \,.\, \cos \overline{|4D+S|} + \overset{E}{{\cdot}0000082} \,.\, \cos \overline{|4D-S|},$$

which gives $\left(\frac{r}{a}\right)^2$, requiring no further change.

Then for $\delta \,.\, \left(\frac{r}{a}\right)^2$

we have,

$$+\frac{\delta E}{E} \times \left\{ + {\cdot}0002763 \,.\, \cos \overline{|S|} + {\cdot}0000150 \,.\, \cos \overline{|2S|} - {\cdot}0000880 \,.\, \cos \overline{|D+S|} \right.$$

$$+ {\cdot}0000021 \,.\, \cos \overline{|D-S|} + {\cdot}0001624 \,.\, \cos \overline{|2D+S|}$$

$$- {\cdot}0010217 \,.\, \cos \overline{|2D-S|} - {\cdot}0000842 \,.\, \cos \overline{|2D-2S|}$$

$$\left. - {\cdot}0000021 \,.\, \cos \overline{|4D+S|} + {\cdot}0000082 \,.\, \cos \overline{|4D-S|} \right\}.$$

Multiplying the expression for

$$\left(\frac{A}{R}\right)^3 + \delta \,.\, \left(\frac{A}{R}\right)^3$$

by that for

$$\left(\frac{r}{a}\right)^2 + \delta \,.\, \left(\frac{r}{a}\right)^2$$

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we obtain the expression for

$$\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^3.\left(\frac{r}{a}\right)^2 + \delta.\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^3.\left(\frac{r}{a}\right)^2,$$

as follows:—

Value of $\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^3.\left(\frac{r}{a}\right)^2$.	Factors of $\frac{\delta\mathrm{E}}{\mathrm{E}}$ in the Variation of $\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^3.\left(\frac{r}{a}\right)^2$. (The first line in each bracket gives the factor depending on $\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^3$; the second gives that depending on $\left(\frac{r}{a}\right)^2$).	
+ 1·0051177	+ ·0008539 + ·0000070	= + ·0008609
+ ·0005407 . cos $\lceil \mathrm{D} \rceil$	− ·0000017 − ·0000021	= − ·0000038 . cos $\lceil \mathrm{D} \rceil$
− ·0148598 . cos $\lceil 2\mathrm{D} \rceil$	− ·0000342 − ·0000217	= − ·0000559 . cos $\lceil 2\mathrm{D} \rceil$
+ ·0000001 . cos $\lceil 4\mathrm{D} \rceil$	+ ·0000001 + ·0000001	= + ·0000002 . cos $\lceil 4\mathrm{D} \rceil$
+ ·0508402 . cos $\lceil \mathrm{S} \rceil$	+ ·0506071 + ·0002770	= + ·0508841 . cos $\lceil \mathrm{S} \rceil$
+ ·0012855 . cos $\lceil 2\mathrm{S} \rceil$	+ ·0025490 + ·0000220	= + ·0025710 . cos $\lceil 2\mathrm{S} \rceil$
+ ·0000314 . cos $\lceil 3\mathrm{S} \rceil$	+ ·0000936 + ·0000006	= + ·0000942 . cos $\lceil 3\mathrm{S} \rceil$
− ·0000743 . cos $\lceil \mathrm{D}+\mathrm{S} \rceil$	+ ·0000136 − ·0000880	= − ·0000744 . cos $\lceil \mathrm{D}+\mathrm{S} \rceil$
+ ·0000158 . cos $\lceil \mathrm{D}-\mathrm{S} \rceil$	+ ·0000135 + ·0000020	= + ·0000155 . cos $\lceil \mathrm{D}-\mathrm{S} \rceil$
− ·0000019 . cos $\lceil \mathrm{D}+2\mathrm{S} \rceil$	− ·0000015 − ·0000022	= − ·0000037 . cos $\lceil \mathrm{D}+2\mathrm{S} \rceil$
+ ·0000004 . cos $\lceil \mathrm{D}-2\mathrm{S} \rceil$	+ ·0000008 + ·0000001	= + ·0000009 . cos $\lceil \mathrm{D}-2\mathrm{S} \rceil$
+ ·0000000 . cos $\lceil \mathrm{D}+3\mathrm{S} \rceil$	− ·0000001 − ·0000001	= − ·0000002 . cos $\lceil \mathrm{D}+3\mathrm{S} \rceil$
− ·0002114 . cos $\lceil 2\mathrm{D}+\mathrm{S} \rceil$	− ·0003747 + ·0001619	= − ·0002128 . cos $\lceil 2\mathrm{D}+\mathrm{S} \rceil$
− ·0013963 . cos $\lceil 2\mathrm{D}-\mathrm{S} \rceil$	− ·0003752 − ·0010241	= − ·0013993 . cos $\lceil 2\mathrm{D}-\mathrm{S} \rceil$
− ·0000053 . cos $\lceil 2\mathrm{D}+2\mathrm{S} \rceil$	− ·0000147 + ·0000041	= − ·0000106 . cos $\lceil 2\mathrm{D}+2\mathrm{S} \rceil$

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Value of $\left(\frac{A}{R}\right)^3 . \left(\frac{r}{a}\right)^2$.	Factors of $\frac{\delta E}{E}$ in the Variation of $\left(\frac{A}{R}\right)^3 . \left(\frac{r}{a}\right)^2$. (The first line in each bracket gives the factor depending on $\left(\frac{A}{R}\right)^3$; the second gives that depending on $\left(\frac{r}{a}\right)^2$.)	
$-\ {\cdot}0000772 \,.\, \cos |\overline{2D-2S}|$	$-{\cdot}0000445$ $-{\cdot}0001099$	$= -{\cdot}0001544 \,.\, \cos |\overline{2D-2S}|$
$-\ {\cdot}0000001 \,.\, \cos |\overline{2D+3S}|$	$-{\cdot}0000005$ $+{\cdot}0000001$	$= -{\cdot}0000004 \,.\, \cos |\overline{2D+3S}|$
$-\ {\cdot}0000020 \,.\, \cos |\overline{2D-3S}|$	$-{\cdot}0000031$ $-{\cdot}0000027$	$= -{\cdot}0000058 \,.\, \cos |\overline{2D-3S}|$
${\cdot}0000000 \,.\, \cos |\overline{2D-4S}|$	$-{\cdot}0000001$ $-{\cdot}0000001$	$= -{\cdot}0000002 \,.\, \cos |\overline{2D-4S}|$
$-\ {\cdot}0000021 \,.\, \cos |\overline{4D+S}|$	${\cdot}0000000$ $-{\cdot}0000021$	$= -{\cdot}0000021 \,.\, \cos |\overline{4D+S}|$
$+\ {\cdot}0000082 \,.\, \cos |\overline{4D-S}|$	${\cdot}0000000$ $+{\cdot}0000082$	$= +{\cdot}0000082 \,.\, \cos |\overline{4D-S}|$
$+\ {\cdot}0000002 \,.\, \cos |\overline{4D-2S}|$	$+{\cdot}0000002$ $+{\cdot}0000002$	$= +{\cdot}0000004 \,.\, \cos |\overline{4D-2S}|$
$-\ {\cdot}0000001 \,.\, \cos |\overline{4D+2S}|$	$-{\cdot}0000001$ $-{\cdot}0000001$	$= -{\cdot}0000002 \,.\, \cos |\overline{4D+2S}|$

Several of the numbers given above can produce no effect in our ultimate result; but it is less difficult and more satisfactory to include all than to make a selection.

(22.) We now proceed with the formation of the quantities which multiply $\left(\frac{A}{R}\right)^3 . \left(\frac{r}{a}\right)^2$; commencing with the argument $|\overline{2(v-V)}|$.

Referring again to Delaunay and Leverrier, and putting nt and Nt for the mean longitudes of the Moon and Sun,

$$\begin{aligned} v = nt &- {\cdot}0006169 \,.\, \sin |\overline{D}| + {\cdot}0114887 \,.\, \sin |\overline{2D}| \\ &- {\cdot}0032422 \,.\, \sin |\overline{S}| - {\cdot}0000363 \,.\, \sin |\overline{2S}| \\ &- {\cdot}0000007 \,.\, \sin |\overline{3S}| - {\cdot}0001192 \,.\, \sin |\overline{2D+S}| \\ &+ {\cdot}0008016 \,.\, \sin |\overline{2D-S}| : \end{aligned}$$

$$V = Nt + 0335409 \,.\, \sin |\overline{S}| + {\cdot}0003515 \,.\, \sin |\overline{2S}| + {\cdot}0000050 \,.\, \sin |\overline{3S}|.$$

Therefore,

$$\begin{aligned} v - V = |\overline{nt - Nt}| &- {\cdot}0006169 \,.\, \sin |\overline{D}| + {\cdot}0114887 \,.\, \sin |\overline{2D}| \\ &- {\cdot}0367831 \,.\, \sin |\overline{S}| - {\cdot}0003878 \,.\, \sin |\overline{2S}| - {\cdot}0000057 \,.\, \sin |\overline{3S}| \\ &- {\cdot}0001192 \,.\, \sin |\overline{2D+S}| + {\cdot}0008016 \,.\, \sin |\overline{2D-S}|. \end{aligned}$$

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But $|\overline{nt - Nt}| = D$. Making this substitution, and doubling the whole;

$$|\overline{2(v-V)}| = \left\{\begin{array}{l} 2D \\ \quad + (-\cdot 0012338 \,.\, \sin |\overline{D}| + \cdot 0229774 \,.\, \sin |\overline{2D}| \\ \quad - \cdot 0735662 \,.\, \sin |\overline{S}| - \cdot 0007756 \,.\, \sin |\overline{2S}| - \cdot 0000114 \,.\, \sin |\overline{3S}| \\ \quad - \cdot 0002384 \,.\, \sin |\overline{2D+S}| + \cdot 0016032 \,.\, \sin |\overline{2D-S}|) \end{array}\right\}$$

We shall call this

$$|\overline{2D}| + \Delta(2D).$$

Then

$$\sin |\overline{2(v-V)}| = \sin |\overline{2D}| \,.\, \cos |\overline{\Delta(2D)}| + \cos |\overline{2D}| \,.\, \sin |\overline{\Delta(2D)}| ;$$

$$\cos |\overline{2(v-V)}| = \cos |\overline{2D}| \,.\, \cos |\overline{\Delta(2D)}| - \sin |\overline{2D}| \,.\, \sin |\overline{\Delta(2D)}| ;$$

and we must now ascertain the values of cos $|\overline{\Delta(2D)}|$ and sin $|\overline{\Delta(2D)}|$.

(23.) The series giving these values are

$$\cos |\overline{\Delta(2D)}| = 1 - \frac{1}{2}\left\{\Delta(2D)\right\}^2 + \frac{1}{24}\left\{\Delta(2D)\right\}^4 - \&c.$$

$$\sin |\overline{\Delta(2D)}| = \Delta(2D) - \frac{1}{6}\left\{\Delta(2D)\right\}^3 + \frac{1}{120}\left\{\Delta(2D)\right\}^5 - \&c.$$

Now, it appears on trial of the principal terms, with due regard of the repeated bisections in combinations of arguments, that $\frac{1}{24}\{\Delta(2D)\}^4$ and higher powers are not sensible for the investigation before us. Therefore we have

$$\sin |\overline{2(v-V)}| = \sin |\overline{2D}| \,.\, \left\{1 - \frac{1}{2}\{\Delta(2D)\}^2\right\}$$
$$+ \cos |\overline{2D}| \,.\, \left\{\Delta(2D) - \frac{1}{6}\{\Delta(2D)\}^3\right\} ;$$

$$\cos |\overline{2(v-V)}| = \cos |\overline{2D}| \,.\, \left\{1 - \frac{1}{2}\{\Delta(2D)\}^2\right\}$$
$$- \sin |\overline{2D}| \,.\, \left\{\Delta(2D) - \frac{1}{6}\{\Delta(2D)\}^3\right\} ;$$

each of which is ultimately to be multiplied by

$$\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 .$$

(24.) For $\{\Delta(2D)\}^2$, I have thought it desirable to form every product as low as ·0000001 (rejecting a few of the smallest); but for $\{\Delta(2D)\}^3$, I have been satisfied with multiplying the four largest terms of $\{\Delta(2D)\}^2$ by the two largest terms of $\Delta(2D)$. Thus we obtain:—

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$\Delta(2D)$			$\{\Delta(2D)\}^2$			$\{(\Delta 2D)\}^3$		
			Non-periodic $=$	$(+\cdot 0029723$	$+\cdot 0054158 \times \frac{\delta E}{E})$			
$+\sin \vert D \vert$	$\times(-\cdot 0012338$	$)$	$+\cos \vert D \vert$	$\times(-\cdot 0000284$	$)$			
$+\sin \vert 2D \vert$	$\times(+\cdot 0229774$	$)$	$+\cos \vert 2D \vert$	$\times(+\cdot 001349$	$+\cdot 0002712 \times \frac{\delta E}{E})$	$+\sin \vert 2D \vert$	$\times(+\cdot 0001926$	$+\cdot 0003716 \times \frac{\delta E}{E})$
			$+\cos \vert 3D \vert$	$\times(+\cdot 0000284$				
			$+\cos \vert 4D \vert$	$\times(-\cdot 0002636$	$+\cdot 0000008 \times \frac{\delta E}{E})$			
$+\sin \vert S \vert$	$\times(-\cdot 0735662$	$-\cdot 0735662 \times \frac{\delta E}{E})$	$+\cos \vert S \vert$	$\times(+\cdot 0000883$	$+\cdot 0002023 \times \frac{\delta E}{E})$	$\times\sin \vert S \vert$	$+(-\cdot 0003753$	$-\cdot 0009502 + \frac{\delta E}{E})$
$+\sin \vert 2S \vert$	$+(-\cdot 0007756$	$-\cdot 0015512 \times \frac{\delta E}{E})$	$+\cos \vert 2S \vert$	$\times(-\cdot 0027056$	$-\cdot 0054096 \times \frac{\delta E}{E})$			
$+\sin \vert 3S \vert$	$\times(-\cdot 0000114$	$\cdot 0000342 \times \frac{\delta E}{E})$	$+\cos \vert 3S \vert$	$\times(-\cdot 0000570$	$-\cdot 0001710 \times \frac{\delta E}{E})$	$\times\sin \vert 3S \vert$	$\times(+\cdot 0000996$	$+\cdot 0002988 \times \frac{\delta E}{E})$
			$+\cos \vert 4S \vert$	$\times(-\cdot 0000011$	$-\cdot 0000044 \times \frac{\delta E}{E})$			
			$+\cos \vert D+S \vert$	$\times(-\cdot 0000906$	$-\cdot 0000906 \times \frac{\delta E}{E})$			

(Continued.)

$\Delta(2D)$	$\{\Delta(2D)\}^2$	$\{\Delta(2D)\}^3$
	$+\cos\,\overline{D-S}\times\left(+{\cdot}0000888\quad+{\cdot}0000888\times\frac{\delta E}{E}\right)$	
$+\sin\,\overline{2D+S}\times\left(-{\cdot}0002384\quad-{\cdot}0002384\times\frac{\delta E}{E}\right)$	$+\cos\,\overline{2D+S}\times\left(+{\cdot}0016915\quad+{\cdot}0016939\times\frac{\delta E}{E}\right)$	
$+\sin\,\overline{2D-S}\times\left(+{\cdot}0016032\quad+{\cdot}0016032\times\frac{\delta E}{E}\right)$	$+\cos\,\overline{2D-S}\times\left(-{\cdot}0016901\quad-{\cdot}0016896\times\frac{\delta E}{E}\right)$	
		$+\sin\,\overline{2D+2S}\times\left(-{\cdot}0000933\quad-{\cdot}0001866\times\frac{\delta E}{E}\right)$
	$+\cos\,\overline{2D-2S}\times\left(-{\cdot}0001358\quad-{\cdot}0002716\times\frac{\delta E}{E}\right)$	$+\sin\,\overline{2D-2S}\times\left(-{\cdot}0000933\quad-{\cdot}0001866\times\frac{\delta E}{E}\right)$
	$+\cos\,\overline{2D-3S}\times\left(-{\cdot}0000014\quad-{\cdot}0000042\times\frac{\delta E}{E}\right)$	
	$+\cos\,\overline{3D-S}\times\left(+{\cdot}0000020\quad+{\cdot}0000020\times\frac{\delta E}{E}\right)$	
	$+\cos\,\overline{4D+S}\times\left(+{\cdot}0000056\quad+{\cdot}0000056\times\frac{\delta E}{E}\right)$	$+\sin\,\overline{4D+S}\times\left(+{\cdot}0000195\quad+{\cdot}0000195\times\frac{\delta E}{E}\right)$
	$+\cos\,\overline{4D-S}\times\left(-{\cdot}0000369\quad-{\cdot}0000369\times\frac{\delta E}{E}\right)$	$+\sin\,\overline{4D-S}\times\left(-{\cdot}0000195\quad-{\cdot}0000195\times\frac{\delta E}{E}\right)$
	$+\cos\,\overline{4D-2S}\times\left(-{\cdot}0000013\quad-{\cdot}0000026\times\frac{\delta E}{E}\right)$	

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(25.) From these we form the following expressions for $\cos\,\overline{|\Delta(2D)|}$ and $\sin\,\overline{|\Delta(2D)|}$, equal respectively to $1-\frac{1}{2}\{\Delta(2D)\}^2$ and $\Delta(2D)-\frac{1}{6}\{\Delta(2D)\}^3$.

$$\cos\,\overline{|\Delta(2D)|}$$

$$\begin{array}{llll}
\text{Non-periodic} = & (+\,{\cdot}9985138 & -\,{\cdot}0027079 \times \frac{\delta E}{E}) \\
+\cos\,\overline{|D|} & \times\,(+\,{\cdot}0000142 &) \\
+\cos\,\overline{|2D|} & \times\,(-\,{\cdot}0000674 & -\,{\cdot}0001356 \times \frac{\delta E}{E}) \\
+\cos\,\overline{|3D|} & \times\,(-\,{\cdot}0000142 &) \\
+\cos\,\overline{|4D|} & \times\,(+\,{\cdot}0001318 & -\,{\cdot}0000004 \times \frac{\delta E}{E}) \\
+\cos\,\overline{|S|} & \times\,(-\,{\cdot}0000442 & -\,{\cdot}0001012 \times \frac{\delta E}{E}) \\
+\cos\,\overline{|2S|} & \times\,(+\,{\cdot}0013528 & +\,{\cdot}0027048 \times \frac{\delta E}{E}) \\
+\cos\,\overline{|3S|} & \times\,(+\,{\cdot}0000285 & +\,{\cdot}0000855 \times \frac{\delta E}{E}) \\
+\cos\,\overline{|4S|} & \times\,(+\,{\cdot}0000006 & +\,{\cdot}0000022 \times \frac{\delta E}{E})
\end{array}$$

$$\sin\,\overline{|\Delta(2D)|}$$

$$\begin{array}{lll}
+\sin\,\overline{|D|} & \times\,(-\,{\cdot}0012338 & \\
+\sin\,\overline{|2D|} & \times\,(+\,{\cdot}0229453 & -\,{\cdot}0000619 \times \frac{\delta E}{E}) \\
+\sin\,\overline{|S|} & \times\,(-\,{\cdot}0735066 & -\,{\cdot}\,734078 \times \frac{\delta E}{E}) \\
+\sin\,\overline{|2S|} & \times\,(-\,{\cdot}0007756 & -\,{\cdot}0015512 \times \frac{\delta E}{E}) \\
+\sin\,\overline{|3S|} & \times\,(-\,{\cdot}0000280 & -\,{\cdot}0000840 \times \frac{\delta E}{E})
\end{array}$$

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(Continued.)

$$\cos\,\overline{\Delta(2D)}$$

$$+\cos\,\overline{D+S}\quad\times\left(+\ {\cdot}0000453\quad+\ {\cdot}0000453\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{D-S}\quad\times\left(-\ {\cdot}0000444\quad-\ {\cdot}0000444\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{2D+S}\quad\times\left(-\ {\cdot}0008458\quad-\ {\cdot}0008470\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{2D-S}\quad\times\left(+\ {\cdot}0008450\quad+\ {\cdot}0008448\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{2D-2S}\quad\times\left(+\ {\cdot}0000679\quad+\ {\cdot}0001358\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{2D-3S}\quad\times\left(+\ {\cdot}0000007\quad+\ {\cdot}0000021\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{3D-S}\quad\times\left(-\ {\cdot}0000010\quad-\ {\cdot}0000010\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{4D+S}\quad\times\left(-\ {\cdot}0000028\quad-\ {\cdot}0000028\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{4D-S}\quad\times\left(+\ {\cdot}0000184\quad+\ {\cdot}0000184\times\frac{\delta E}{E}\right)$$

$$+\cos\,\overline{4D-2S}\quad\times\left(+\ {\cdot}0000006\quad+\ {\cdot}0000013\times\frac{\delta E}{E}\right)$$

$$\sin\,\overline{\Delta(2D)}$$

$$+\sin\,\overline{2D+S}\quad\times\left(-\ {\cdot}0002384\quad-\ {\cdot}0002384\times\frac{\delta E}{E}\right)$$

$$+\sin\,\overline{2D-S}\quad\times\left(+\ {\cdot}0016032\quad+\ {\cdot}0016032\times\frac{\delta E}{E}\right)$$

$$+\sin\,\overline{2D+2S}\quad\times\left(+\ {\cdot}0000156\quad+\ {\cdot}0000311\times\frac{\delta E}{E}\right)$$

$$+\sin\,\overline{2D-2S}\quad\times\left(+\ {\cdot}0000156\quad+\ {\cdot}0000311\times\frac{\delta E}{E}\right)$$

$$+\sin\,\overline{4D+S}\quad\times\left(-\ {\cdot}0000033\quad-\ {\cdot}0000033\times\frac{\delta E}{E}\right)$$

$$+\sin\,\overline{4D-S}\quad\times\left(+\ {\cdot}0000033\quad+\ {\cdot}0000033\times\frac{\delta E}{E}\right)$$

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(26.) We are now to combine the expressions last found, in order to produce the quantities

$$T\,.\,\frac{1}{a}\,.\,\frac{r}{a}\quad\text{and}\quad P\,.\,\frac{1}{a}\,.\,\frac{r}{a},$$

or

$$-{\cdot}0083928\times\left(\frac{A}{R}\right)^3\cdot\left(\frac{r}{a}\right)^2\times\sin\left|\overline{2(v-V)}\right|,$$

$$+{\cdot}0083928\times\left(\frac{A}{R}\right)^3\cdot\left(\frac{r}{a}\right)^2\times\left\{+\frac{1}{3}+\cos\left|\overline{2(v-V)}\right|\right\}$$

(of which we are to retain only the terms that multiply $\frac{\delta E}{E}$, or, as we may conveniently express it, $E'.t$, where E' is a constant), and to examine the results of substituting the combinations in the Factorial Equations. Our ultimate object is, not to retain every term in the solution of the equations, but only the non-periodic terms (increasing with the time) of the longitude and the radius vector, especially the longitude (which alone will ultimately possess any value for us). And this consideration is to be kept in view in solving the factorial equations. We will now consider separately the two disturbing functions.

(27.) On examining the formation of the selected terms of

$$\left(\frac{A}{R}\right)^3\cdot\left(\frac{r}{a}\right)^2\times\sin\left|\overline{2(v-V)}\right|,$$

or

$$\left(\left(\frac{A}{R}\right)^3\cdot\left(\frac{r}{a}\right)^2\times\cos\left|\overline{\Delta(2D)}\right|\right)\times\sin\left|\overline{2D}\right|$$

$$+\left(\left(\frac{A}{R}\right)^3\cdot\left(\frac{r}{a}\right)^2\times\sin\left|\overline{\Delta(2D)}\right|\right)\times\cos\left|\overline{2D}\right|,$$

with reference to the expansions of these functions lately exhibited, it appears that the first part is the product of a constant (accompanied with a series of cosines) by a sine, and the second is the product of a series of sines by a cosine. Each of these products will be a series of sines. In order to ascertain whether these will produce any constant term in the ultimate result, it is necessary to refer to the factorial equations. Putting

$$Ft\,.\,\sin\left|\overline{Gt}\right|$$

for one of the terms of the expansion of T, considered alone (which is legitimate, because the equations are linear), we have the factorial equations,

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$$
Ft\,.\sin\left|\overline{Gt}\right| = \left\{\begin{array}{lll}
+\delta\left(\dfrac{a}{r}\right) & \times\{ & {\cdot}0000 - {\cdot}0281\,.\sin\left|\overline{2D}\right|\} \\
+\dfrac{d}{dt}\left(\delta\dfrac{a}{r}\right) & \times\{ & -1{\cdot}9919 + {\cdot}0061\,.\cos\left|\overline{2D}\right|\} \\
+\delta v & \times\{ & {\cdot}0000 + {\cdot}0168\,.\cos\left|\overline{2D}\right|\} \\
+\dfrac{d}{dt}(\delta v) & \times\{ & {\cdot}0000 + {\cdot}0268\,.\sin\left|\overline{2D}\right|\} \\
+\dfrac{d^2}{dt^2}(\delta v) & \times\{ & +1{\cdot}0006 - {\cdot}0144\,.\cos\left|\overline{2D}\right|\}
\end{array}\right\}.
$$

$$
0 = \left\{\begin{array}{lll}
+\delta\left(\dfrac{a}{r}\right) & \times\{ & +2{\cdot}9964 - {\cdot}0236\,.\cos\left|\overline{2D}\right|\} \\
+\dfrac{d}{dt}\left(\delta\dfrac{a}{r}\right) & \times\{ & {\cdot}0000 - {\cdot}0528\,.\sin\left|\overline{2D}\right|\} \\
+\dfrac{d^2}{dt^2}\left(\delta\dfrac{a}{r}\right) & \times\{ & -1{\cdot}0053 + {\cdot}0212\,.\cos\left|\overline{2D}\right|\} \\
+\delta v & \times\{ & {\cdot}0000 + {\cdot}0168\,.\sin\left|\overline{2D}\right|\} \\
+\dfrac{d}{dt}(\delta v) & \times\{ & -1{\cdot}9888 - {\cdot}0090\cdot\cos\left|\overline{2D}\right|\}
\end{array}\right\}.
$$

Using the symbols

$$
\Phi = -{\cdot}0281\,.\,\delta\left(\frac{a}{r}\right).\sin\left|\overline{2D}\right| + {\cdot}0061\,.\frac{d}{dt}\left(\delta\frac{a}{r}\right).\cos\left|\overline{2D}\right| + {\cdot}0168\,.\,\delta v\,.\cos\left|\overline{2D}\right|
$$

$$
+{\cdot}0268\,.\frac{d}{dt}(\delta v)\,.\sin 2D - {\cdot}0144\,.\frac{d^2}{dt^2}(\delta v)\,.\cos\left|\overline{2D}\right|;
$$

$$
\Psi = -{\cdot}0236\,.\,\delta\left(\frac{a}{r}\right).\cos\left|\overline{2D}\right| - {\cdot}0528\,.\frac{d}{dt}\left(\delta\frac{a}{r}\right).\sin\left|\overline{2D}\right|
$$

$$
+{\cdot}0212\,.\frac{d^2}{dt^2}\left(\delta\frac{a}{r}\right).\cos\left|\overline{2D}\right| + {\cdot}0168\,.\,\delta v\,.\sin\left|\overline{2D}\right| - {\cdot}0090\,.\frac{d}{dt}(\delta v).\cos\left|\overline{2D}\right|
$$

the equations may be conveniently written,

$$
\begin{array}{lll}
-1{\cdot}9919\,.\dfrac{d}{dt}\left(\delta\dfrac{a}{r}\right) + 1{\cdot}0006\,.\dfrac{d^2}{dt^2}(\delta v) & = Ft\,.\sin\left|\overline{Gt}\right| & -\Phi; \\
+2{\cdot}9964\,.\,\delta\left(\dfrac{a}{r}\right) - 1{\cdot}0053\,.\dfrac{d^2}{dt^2}\left(\delta\dfrac{a}{r}\right) - 1{\cdot}9888\,.\dfrac{d}{dt}(\delta v) & = \quad 0 & -\Psi
\end{array}
$$

(in which we shall afterwards use, instead of the fractional factors, the integers -2, $+1$, $+3$, -1, -2); and these equations will be most conveniently solved by successive substitution; first, neglecting the small quantities Φ and Ψ; then solving the equations so limited, substituting these limited solutions in Φ and Ψ; and then re-solving; and so on. It is easily found that limited solutions are

$$
\delta\left(\frac{a}{r}\right) = \frac{2G}{G^2 - G^4}\,Ft\,.\cos\left|\overline{Gt}\right|, \qquad \delta v = \frac{3 + G^2}{G^2 - G^4}\,Ft\,.\sin\left|\overline{Gt}\right|;
$$

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and these, on substitution, destroy the term whose factor is Ft, but introduce other terms. Upon making the substitutions

$$\delta\left(\frac{a}{r}\right) = \frac{2G}{G^2-G^4}Ft\,.\cos\overline{\vert Gt\vert} + c, \qquad \delta v = \frac{3+G^2}{G^2-G^4}Ft\,.\sin\overline{\vert Gt\vert} + s,$$

we obtain the following equations,

$$-2\,.\frac{dc}{dt} + \frac{d^2s}{dt^2} = -\frac{2G+2G^3}{G^2-G^4}F\,.\cos\overline{\vert Gt\vert} - \Phi;$$

$$+3.c - \frac{d^2c}{dt^2} - 2.\frac{ds}{dt} = +\frac{6-2G^2}{G^2-G^4}\quad F\,.\sin\overline{\vert Gt\vert} - \Psi;$$

and, on substituting in Φ and Ψ the first terms of the expressions for $\delta\left(\frac{a}{r}\right)$, δv, and their differentials, we find equations exactly similar to those whose solution is given in Articles 11 and 12. The precise values of the coefficients for $\delta\frac{a}{r}$ and δv are of no importance; at present it is only necessary for us to remark that they are constants for each value of G.

We shall now consider the character of the expressions for $\delta\frac{a}{r}$ and δv: first, for the general value of G; secondly for the the special value $Gt = 2D$.

For the general case, we are to substitute in Φ and Ψ the very approximate values of $\delta\left(\frac{a}{r}\right)$ and δv given by the formulæ

$$\delta\left(\frac{a}{r}\right) = Ct\,.\cos\overline{\vert Gt\vert}, \quad \delta v = St\,.\sin\overline{\vert Gt\vert};$$

and we find

$$\Phi = \left\{\begin{array}{ll} +\{+\cdot0281\,.C \;-\cdot0268\,.GS\} & \times t\,.\sin\overline{\vert 2D\vert}\,.\cos\overline{\vert Gt\vert} \\ +\{+\cdot0061\,.GC-\cdot0144\,.G^2S-\cdot0168\,.S\} & \times t\,.\cos\overline{\vert 2D\vert}\,.\sin\overline{\vert Gt\vert} \\ +\{-\cdot0061\,.C \;+\cdot0288\,.GS\} & \times \;\cos\overline{\vert 2D\vert}\,.\cos\overline{\vert Gt\vert} \\ +\{-\cdot0268\,.S\} & \times \;\sin\overline{\vert 2D\vert}\,.\sin\overline{\vert Gt\vert} \end{array}\right\}$$

$$\Psi = \left\{\begin{array}{ll} +\{+\cdot0236\,.C \;+\cdot0212\,.G^2C+\cdot0090\,.GS\} & \times t\,.\cos\overline{\vert 2D\vert}\,.\cos\overline{\vert Gt\vert} \\ +\{-\cdot0528\,.GC-\cdot0168\,.S\} & \times t\,.\sin\overline{\vert 2D\vert}\,.\sin\overline{\vert Gt\vert} \\ +\{+\cdot0528\,.C\} & \times \;\sin\overline{\vert 2D\vert}\,.\cos\overline{\vert Gt\vert} \\ +\{+\cdot0424\,.C \;+\cdot0090\,.S\} & \times \;\cos\overline{\vert 2D\vert}\,.\sin\overline{\vert Gt\vert} \end{array}\right\}$$

Now, all these terms, as well as the two new terms in the equation lately formed, are periodical; and the form of the equations for c and s is the same as that for $\delta\left(\frac{a}{r}\right)$ and δv; and

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therefore, to this second step of approximation as well as in the first, the value of δv contains no non-periodical factor of t. And, as far as can be seen, the same remark applies to succeeding approximations.

For the case when $Gt = 2D$ (which in Article 9 and succeeding Articles is written F) we have only to remark that the equations before us are already solved in Articles 9 and 12. For, as the treatment of A, and of B, and of C, in those equations, is absolutely independent, we have merely to extract (for the effect of the term $-A \,.\, bt \,.\, \sin\overline{2D}$ in Article 9) the multiples of Ab in Article 12. And, referring to the terms in Article 11 of which g and l are the factors, it will be seen that those terms are periodical.

Therefore the term

$$-\cdot 0083928 \times \left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \sin\overline{2(v-V)}$$

produces no non-periodical term in δv.

(28.) On examining the formation of the terms of

$$\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \cos\overline{2(v-V)},$$

it will be seen that they consist of two parts. One is the series of constants produced by the products of similar cosines or similar sines, to be attached to a term of the same form in

$$+\cdot 0083928 \times \left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \frac{1}{3},$$

of which we shall treat shortly. The other part is a series of products of cosines or sines of different arguments, producing cosines of new arguments.

For the general term $\cos\overline{Gt}$, in the part just mentioned, the same reasoning will apply as that which has been employed for $\sin\overline{Gt}$; the expressions for Φ and Ψ being the same as those which are used in discussing the result of $\sin\overline{Gt}$; and the same conclusion following—namely, that there is no non-periodic term.

In the special case when $Gt = 2D$, the solution has already been effected in Articles 9 and 12, in treating the factor C. And the result is, that it produces no non-periodical term.

Therefore the term

$$+\cdot 0083928 \times \cos\overline{2(v-V)}$$

produces no non-periodic term, except from those non-periodic parts which we shall associate with the non-periodic parts of

$$+\cdot 0083928 \times \left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \frac{1}{3}.$$

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(29.) For the non-periodic parts, whatever be their origin, we have again to remark that the solution is given in Articles 9 and 12; where the term $Bb.t$ of Article 9 produces, in Article 12, for $\delta\left(\frac{a}{r}\right)$ the term $-Bbt$, and for δv the term $-Bbt^2$.

(30.) The general result of these investigations is, that we are merely to extract, from the details of the two products

$$+\cdot 0083928 \times \left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \frac{1}{3},$$

and

$$+\cdot 0083928 \times \left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \cos \overline{|2(v-V)|},$$

the factors of $\frac{\delta E}{E}$ or $E't$; and to use them in forming the Factorial Equations. To this extraction from details we shall now give our attention.

(31.) Referring to the expansion in Article 22, it will appear that we must consider these three terms:

$$+\frac{1}{3}\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2,$$

$$+\left(\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \cos \overline{|\Delta(2D)|}\right) \times \cos \overline{|2D|},$$

$$-\left(\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2 \times \sin \overline{|\Delta(2D)|}\right) \times \sin \overline{|2D|}.$$

The first part is merely $\frac{1}{3}$ of the development in Article 21, and consists of a constant and a series of cosines.

The second part consists of the product of a constant and a series of cosines by

$$\cos \overline{|2D|},$$

The third part consists of the product of a series of sines by

$$\sin \overline{|2D|},$$

Each of these parts will give rise to non-periodic terms.

(32.) For development of the first part, we have merely to take one-third of the non-periodic term in Article 21. Here, and in the development of the following parts, we shall retain only the multiples of $\frac{\delta E}{E}$. Thus we obtain for the first part:

$$+\frac{\delta E}{E} \times +\cdot 0002870.$$

For the second and third parts we shall proceed to form the products to which allusion is made in the last sentences of Article 26. The only arguments in

$$\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2$$

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which can combine with those of

$$\cos \lceil \Delta(2D) \rceil \quad \text{and} \quad \sin \lceil \Delta(2D) \rceil$$

to form

$$\cos \lceil 2D \rceil \quad \text{or} \quad \sin \lceil 2D \rceil$$

are the following:

	Term of $\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2$.		Term of cos $\lceil \Delta(2D) \rceil$ or sin $\lceil \Delta(2D) \rceil$.
*	Non-periodic	will combine with	$\lceil 2D \rceil$
*	$\lceil D \rceil$	„	$\lceil D \rceil$
	$\lceil D \rceil$	„	$\lceil 3D \rceil$
*	$\lceil 2D \rceil$	„	non-periodic
*	$\lceil 2D \rceil$	„	$\lceil 4D \rceil$
	$\lceil 4D \rceil$	„	$\lceil 2D \rceil$
*	$\lceil S \rceil$	„	$\lceil 2D+S \rceil$
*	$\lceil S \rceil$	„	$\lceil 2D-S \rceil$
	$\lceil 2S \rceil$	„	$\lceil 2D+2S \rceil$
*	$\lceil 2S \rceil$	„	$\lceil 2D-2S \rceil$
	$\lceil 3S \rceil$	„	$\lceil 2D-3S \rceil$
	$\lceil D+S \rceil$	„	$\lceil D-S \rceil$
	$\lceil D-S \rceil$	„ . .	$\lceil D+S \rceil$
	$\lceil D-S \rceil$	„	$\lceil 3D-S \rceil$
*	$\lceil 2D-S \rceil$	„	$\lceil S \rceil$
	$\lceil 2D-S \rceil$	„	$\lceil 4D-S \rceil$
*	$\lceil 2D-2S \rceil$	„	$\lceil 2S \rceil$
	$\lceil 2D-2S \rceil$	„	$\lceil 4D-2S \rceil$

Only those combinations which are distinguished by an asterisk can produce numbers included in the first seven decimals.

(33.) We now exhibit the products. It will be remembered that (except where one factor is non-periodic) each multiplication for cos $\lceil \Delta(2D) \rceil$ produces two cosines, of which we shall retain only one, namely, cos $\lceil 2D \rceil$, and that its coefficient is to be only one-half of the product of the coefficients which form it. In like manner each multiplication for sin $\lceil \Delta(2D) \rceil$ produces two sines, of which we retain only sin $\lceil 2D \rceil$.

Also, that we retain only the factors of $\frac{\delta E}{E}$.

The following are the special multiplications:—

Products giving the factors of $\cos \boxed{2D} \cdot \frac{\delta E}{E}$, to be subsequently multiplied by $+ \cos \boxed{2D}$.

	Term of $\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2$			Term of $\cos \boxed{\Delta(2D)}$		Product multiplying $\cos \boxed{2D} \cdot \frac{\delta E}{E}$
Non-periodic	$(+1{\cdot}0051177$	$+{\cdot}0008609 \times \frac{\delta E}{E}) \times$	$+ \cos \boxed{2D}$	$\times(-{\cdot}0000674$	$-{\cdot}0001356 \times \frac{\delta E}{E})$	$-{\cdot}0001363$ 0000000
$+ \cos \boxed{2D}$	$\times(-\ {\cdot}0148598$	$-{\cdot}0000559 \times \frac{\delta E}{E}) \times$	Non-periodic	$(+{\cdot}9985138$	$-{\cdot}0027079 \times \frac{\delta E}{E})$	$+{\cdot}0000402$ $-{\cdot}0000557$
$+ \cos \boxed{S}$	$\times(+\ {\cdot}0508402$	$+{\cdot}0508841 \times \frac{\delta E}{E}) \times$	$+ \cos \boxed{2D+S}$	$\times(-{\cdot}0008458$	$-{\cdot}0008470 \times \frac{\delta E}{E})$	$-{\cdot}0000431$ $-{\cdot}0000430$
$+ \cos \boxed{S}$	$\times(+\ {\cdot}0508402$	$+{\cdot}0508841 \times \frac{\delta E}{E}) \times$	$+ \cos \boxed{2D-S}$	$\times(+{\cdot}0008450$	$+{\cdot}0008448 \times \frac{\delta E}{E})$	$+{\cdot}0000429$ $+{\cdot}0000430$
$+ \cos \boxed{2S}$	$\times(+\ {\cdot}0012855$	$+{\cdot}0025710 \times \frac{\delta E}{E}) \times$	$+ \cos \boxed{2D-2S}$	$\times(+{\cdot}0000679$	$+{\cdot}0001358 \times \frac{\delta E}{E})$	$+{\cdot}0000001$ $+{\cdot}0000002$
$+ \cos \boxed{2D-S}$	$\times(-\ {\cdot}0013963$	$-{\cdot}0013993 \times \frac{\delta E}{E}) \times$	$+ \cos \boxed{S}$	$\times(-{\cdot}0000442$	$-{\cdot}0001012 \times \frac{\delta E}{E})$	$+{\cdot}0000001$ $+{\cdot}0000001$
$+ \cos \boxed{2D-2S}$	$\times(-\ {\cdot}0000772$	$-{\cdot}0001544 \times \frac{\delta E}{E}) \times$	$+ \cos \boxed{2S}$	$\times(+{\cdot}0013528$	$+{\cdot}0027048 \times \frac{\delta E}{E})$	$-{\cdot}0000002$ $-{\cdot}0000002$

All products, except those for the two first lines, are to be divided by 2.

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PRODUCTS giving the factors of $\sin \overline{2D|} \cdot \frac{\delta E}{E}$, to be subsequently multiplied by $-\sin \overline{2D|}$.

	Term of $\left(\frac{A}{R}\right)^3 \cdot \left(\frac{r}{a}\right)^2$			Term of $\sin \overline{\Delta(2D)|}$		Product multiplying $\sin \overline{2D|} \cdot \frac{\delta E}{E}$
Non-periodic	$(+1·0051177$	$+·0008609 \times \frac{\delta E}{E}) \times$	$+\sin \overline{2D|}$	$\times(+·0229453$	$-·0000619 \times \frac{\delta E}{E})$	$-·0000622$ $+·0000197$
$+\cos \overline{S|}$	$\times(+·0508402$	$+·0508841 \times \frac{\delta E}{E}) \times$	$+\sin \overline{2D+S|}$	$\times(-·0002384$	$-·0002384 \times \frac{\delta E}{E})$	$-·0000121$ $-·0000121$
$+\cos \overline{S|}$	$\times(+·0508402$	$+·0508841 \times \frac{\delta E}{E}) \times$	$+\sin \overline{2D-S|}$	$\times(+·0016032$	$+·0016032 \times \frac{\delta E}{E})$	$+·0000815$ $+·0000816$
$+\cos \overline{2D-S|}$	$\times(-·0013963$	$-·0013993 \times \frac{\delta E}{E}) \times$	$+\sin \overline{S|}$	$\times(-·0735066$	$-·0734078 \times \frac{\delta E}{E})$	$+·0001025$ $+·0001029$
$+\cos \overline{2D-2S|}$	$\times(-·0000772$	$-·0001544 \times \frac{\delta E}{E}) \times$	$+\sin \overline{2S|}$	$\times(-·0007756$	$-·0015512 \times \frac{\delta E}{E})$	$+·0000001$ $+·0000001$

All products, except that for the first line, are to be divided by 2.

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(34.) The non-periodic terms in the three different parts of the effective perturbation-term

$$\text{P, or } + \cdot 0083928 \times \left(\frac{\text{A}}{\text{R}}\right)^3 . \left(\frac{r}{a}\right)^2 \times \left\{\frac{1}{3} + \cos \boxed{2(v-\text{V})}\right\}$$

may now be collected; and the numerical term to be inserted in the Factorial Equations may be formed.

The first part, or

$$+\frac{1}{3} \times \cdot 0083928 \times \left(\frac{\text{A}}{\text{R}}\right)^3 . \left(\frac{r}{a}\right)^2,$$

gives

$$+ \cdot 0083928 \times + \cdot 0002870 \times \frac{\delta\text{E}}{\text{E}}.$$

For the second part, the summation of terms in the last table but one, performed in accordance with the rule at the bottom of the table, gives

$$- \cdot 0001518 \times \cos \boxed{2\text{D}} \times \frac{\delta\text{E}}{\text{E}}.$$

This is to be multiplied by

$$+ \cos \boxed{2\text{D}},$$

which gives for its non-periodic part

$$- \cdot 0000759 \times \frac{\delta\text{E}}{\text{E}}.$$

And the second part becomes

$$+ \cdot 0083928 \times - \cdot 0000759 \times \frac{\delta\text{E}}{\text{E}}.$$

For the third part, the summation of terms in the last table, performed in like manner, gives

$$+ \cdot 0001298 \times \sin \boxed{2\text{D}} \times \frac{\delta\text{E}}{\text{E}}.$$

This is to be multiplied by

$$- \sin \boxed{2\text{D}},$$

producing the non-periodic term

$$- \cdot 0000649 \times \frac{\delta\text{E}}{\text{E}}.$$

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And the third part becomes

$$+\cdot0083928 \times -\cdot0000649 \times \frac{\delta E}{E}.$$

The total value of P is thus

$$+\cdot0083928 \times +\cdot0001462 \times \frac{\delta E}{E}.$$

Now, by Le Verrier's Elements,

$$\frac{\text{variation of E for one year}}{E} = -\frac{\cdot000\,000\,4338}{\cdot016\,769\,27}.$$

And, as one year = 84 units of time (Article 3),

$$\frac{\delta E}{E} = -\frac{1}{84} \times \frac{\cdot000\,000\,4338}{\cdot016\,769\,27} \times t.$$

And therefore the total value of P is

$$-\cdot0083928 \times \cdot0001462 \times \frac{1}{84} \times \frac{\cdot000\,000\,4338}{\cdot016\,769\,27} \times t.$$

For the moment, we shall call this quantity $+Kt$.

(35.) An inspection of the operations in the former part of this paper (Articles 9 to 12) will show that the non-periodical part of the solution of the Factorial Equations is in no way affected by the periodical terms of P and T; and also that the periodical parts there retained in the expressions of the factors are unnecessary. And the use of integral numbers, instead of the fractional numbers in the coefficients entering into those factors, produces only unimportant error in the final result. With these simplifications, our equations become

$$\text{T or} \quad 0 = -2 \,.\, \frac{d}{dt}\left(\delta\frac{a}{r}\right) + \frac{d^2}{dt^2}(\delta v).$$

$$\text{P or} \quad +Kt = +3 \,.\, \delta\left(\frac{a}{r}\right) - \frac{d^2}{dt^2}\left(\delta\frac{a}{r}\right) - 2 \,.\, \frac{d}{dt}(\delta v).$$

It will easily be seen that the assumptions

$$\delta\frac{a}{r} = +Ct, \quad \delta v = +Ht^2,$$

are proper for the solution. Substituting

$$0 = -2C + 2H; \quad Kt = +3 \,.\, Ct - 4 \,.\, Ht;$$

from which

$$H = -K, \quad C = -K;$$

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and the value of Ht^2, which is the 'Acceleration' that we are seeking, is $-Kt^2$. To compute the value corresponding to one year, we must make $t = 84$. The value of 'Acceleration' thus found, like all other angular measures here treated, is expressed in terms of radius; to reduce it to seconds, we must divide it by the length of $1''$ in terms of radius $= \cdot 000004848137$. Thus finally we obtain for Acceleration in longitude for one year, expressed in seconds of arc,

$$+\cdot 0083928 \times \cdot 0001462 \times 84 \times \frac{\cdot 0000004338}{\cdot 01676927} \times \frac{1}{\cdot 000004848137};$$

$$= +0'' \cdot 00054997:$$

and for Acceleration in longitude for 100 years,

$$+5'' \cdot 4997.$$

(36.) The factor $\cos^2 |I|$, which enters into the original expressions for perturbing forces, has not hitherto been taken into account. Its mean value is ·99597; the Acceleration for one year is $+0'' \cdot 00054773$; and finally,

The Acceleration in longitude for 100 years $= 5'' \cdot 4773$.

I commenced this investigation without intention of mentioning in it any name. But I cannot terminate it without offering to Professor Adams my very hearty congratulations on his success in making a correction so large to a theory so important.

I think that the Lunar Theory is now placed in a difficult position. With the elements formerly received, the ancient eclipses were very well explained. With the modified theory, the agreement cannot be so good, and perhaps is impossible.

I am unwilling to abandon the interpretation of the ancient eclipses, and I think that reconciliation must be sought, in some new secular term, or in some alteration of the mean motions either of the Sun or of the Moon.

I apologise to the Society for two errors in the first part of the paper. Article 16 is totally erroneous, and Article 17 contains a numerical error. Both arose from the same circumstance, the haste in which the paper was closed for the Meeting of the Society.

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